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Adaptive Control of Nonlinear Attitude Motions Realizing Linear Closed Loop Dynamics

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An adaptive attitude control law is presented to realize linear closed loop dynamics in the attitude error vector. The Modified Rodrigues Parameters (MRPs) are used along with their associated shadow set as the kinematic variables since they form a nonsingular set for all possible rotations. The desired linear closed loop dynamics can be of either PD or PID form. Only a crude estimate of the moment of inertia matrix is assumed to be known. A nonlinear control law is developed which yields linear closed loop dynamics in terms of the MRPs. An adaptive control law is then developed that enforces these desired linear closed loop dynamics in the presence of large inertia and external disturbance model errors. Because the unforced closed loop dynamics are nominally linear, standard linear control methodologies such as pole placement can be employed to satisfy design requirements such as control bandwidth. The adaptive control law is shown to track the desired linear performance asymptotically without requiring apriori knowledge of either the inertia matrix or external disturbance.

Introduction

WHILE the traditional approach to attitude control is based on linear control theory, recent efforts by several authors indicate a shift toward nonlinear control methods. For example, Wie and Barba[1] and Wei et al.[2] develop the rotational equations of motion using the redundant set of Euler parameters. In contrast, Dwyer[3,4] outlines an approach based on a minimal set of three Euler parameters wherein a nonlinear transformation maps the complete equations of motion into a locally valid linear model that may encounter singular attitudes. It is a well known fact that every three-parameter attitude representation has the problem of singularities. The work of Slotine and Li based on Euler angles also has the same limitation.[5]

To address the problem of singular orientations while using a minimal set of three rigid body attitude coordinates, more recently the Modified Rodrigues Parameters (MRPs) have been proposed. Any rigid body orientation can be described through two numerically distinct sets of MRPs which abide by the same differential kinematic equation. By switching between the original and alternate sets of MRPs (also referred to as the shadow set), it is possible to achieve a globally nonsingular attitude parameterization for all possible ± 360 deg rotations.[6–9]

Given this advantage, there have been several recent attitude control applications employing MRPs as rotational kinematic variables.[10–13] A common feature within all these efforts and other developments by Wen and Kruetz-Delgado[14], Wen et al.,[15] Meyer,[16,17] Reyhanoglu et al.,[18] and Slotine and Li[5] is the control law that is based on a stability analysis driven by an associated Lyapunov analysis. Although such attitude feedback control laws can be found by first defining a candidate Lyapunov function and then extracting the corresponding stabilizing nonlinear control, certain very important concepts from linear control theory, such as closed loop damping and bandwidth, are not very well defined because the corresponding closed loop dynamics are generally nonlinear. To achieve a desired closed loop behavior, the closed loop dynamics are linearized about a reference motion in order to use linear control theory techniques to pick the feedback gains. Depending on the nonlinearity of the exact closed loop equations of motion, the desired closed loop performance will be achieved only in a local neighborhood and not globally.

Instead of first finding a feedback control law and then analyzing the closed loop dynamics stability, it is possible to start out instead with a desired (or prescribed) set of stable closed loop dynamics and then extract the corresponding nonlinear control law using a feedback linearization approach[4,14] common in robotics path planning problems. For example, the closed loop dynamics could be a stable linear differential equation. This technique is very general and can be applied to a multitude of systems. However, depending on the nonlinearity of the dynamical system, the nonlinear control laws extracted from such a feedback linearization approach can be potentially very complex. Paielli and Bach[19] present such an attitude control law derived in terms of the Euler parameter components, and that law is remarkably simple. Compared to standard Lyapunov function derived attitude control laws, their control law expression is only slightly more complex. Further, Paielli and Bach illustrate that this type of control law is rather robust for attitude control problems. However, this control law feeds back the Gibbs vector[7] as an attitude measure that is singular at ± 180 deg (error) rotations about any axis. As an important contribution of this paper, we develop a feedback linearizing control law based on the MRP vector that achieves the desired set of stable closed loop trajectories without encountering singular orientations. This paper also addresses the issue of uncertainty in the moment of inertia matrix. Even if the attitude control law (based on some nominal value of inertia) is robust with respect to inertia uncertainties, the closed loop dynamics will no longer exhibit the desired performance if an incorrect inertia matrix is used in the feedback control law. While the inertia matrix is assumed to be essentially unknown in this development, the goal is to ensure that the feedback control law would still produce the desired closed loop dynamics. To accomplish this task, time-varying update laws for the feedback gain matrices are developed that ensure stability for the dynamics of the system adaptively. While classical adaptive control theory due to Narendra[20] and Sastry[21] has also been employed

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in attitude control problems previously,[1,11,22–24] the present MRP vector-based feedback linearization approach is unique in the sense that it explicitly enables linear closed loop dynamics to be chosen and motivated by useful physical concepts such as damping ratio and loop bandwidth.

The paper first develops all the theory necessary to develop the inverse dynamics approach to obtaining stable closed loop rigid body dynamics. In particular, the MRPs are chosen as the attitude parameters. An adaptive control law is presented that includes an integral feedback term in the desired closed loop dynamics and achieves asymptotic stability even in the presence of a class of unmodeled external disturbances. These results are illustrated through various numerical simulations.

Linear Closed Loop Dynamics

The MRP vector $\boldsymbol{\sigma}$ is adopted as a rigid body attitude measure relative to the target attitude. Note that the vector $\boldsymbol{\sigma}$ contains information about both the principal rotation axis $\hat{\boldsymbol{e}}$ and the principal rotation angle Φ because they are related through

$$\boldsymbol{\sigma} = \hat{\boldsymbol{e}} \tan(\Phi/4) \tag{1}$$

Therefore, if $\boldsymbol{\sigma} \to 0$, then the orientation has returned back to the origin. As a complete revolution is performed (i.e. $\Phi \to 360$ deg), this particular MRP set goes singular. As is shown in Refs. 6 and 8, it is possible to map the original MRP vector $\boldsymbol{\sigma}$ to its corresponding shadow counterpart $\boldsymbol{\sigma}^S$ through

$$\boldsymbol{\sigma}^S = -(1/\sigma^2)\boldsymbol{\sigma} \tag{2}$$

where the notation $\sigma^2 = \boldsymbol{\sigma}^T\boldsymbol{\sigma}$ is used. By choosing to switch the MRPs whenever $\sigma^2 > 1$, the MRP vector remains bounded within a unit sphere. Note that there exists no theoretical restriction that MRP vector switching should take place only on the surface of the three-dimensional unit sphere. Switching when the $\sigma^2 = 1$ surface is penetrated results in the corresponding MRPs always indicating the shortest rotational distance back to the origin.[6,25]

Let's assume that we desire the closed loop dynamics to have the following prescribed *linear* form

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} = 0 \tag{3}$$

where the scalars P and K are the positive velocity and position feedback gains. Observe that both P and K could be chosen to be symmetric, positive definite matrices. However, doing so greatly complicates the resulting algebra. Note that this differential equation only contains kinematic quantities, and there is no explicit dependence on system properties such as inertia. Linear control theory states that, for any initial $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\sigma}}$ vectors, the resulting motion is asymptotically stable. If desired, one could also easily add an integral feedback term with an appropriately chosen gain value K_i to the desired closed loop equations and still retain asymptotic stability

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} + K_i\int_0^t \boldsymbol{\sigma}\,\mathrm{d}t = 0 \tag{4}$$

Note that instead of the MRP vector $\boldsymbol{\sigma}$, any attitude or position vector could have been used. In particular, Paielli and Bach chose to express their linear closed loop equations in terms of the vector components of the Euler parameters.[19]

Let the vector $\boldsymbol{u}$ be an external control torque vector which is applied to a rigid body with the inertia matrix $[I]$. The vector $\boldsymbol{F}_e$ is the unmodeled torque vector due to influences such as atmospheric or solar drag or bearing friction. The vector $\boldsymbol{\omega}$ is the body angular velocity vector. Euler's rotational equations of motion state that

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u} + \boldsymbol{F}_e \tag{5}$$

where the tilde matrix $[\tilde{\boldsymbol{\omega}}]$ is the vector cross product operator defined as

$$[\tilde{\boldsymbol{\omega}}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{6}$$

It is desired to find a nonlinear control law $\boldsymbol{u}$ that will render the closed loop dynamics to be of the stable form in Eq. (3) or (4), assuming the system inertia matrix is perfectly known. To achieve this, we treat the body angular acceleration vector $\dot{\boldsymbol{\omega}}$ as the control variable in the following development. Once the necessary vector $\dot{\boldsymbol{\omega}}$ is found, then the physical control torque is found through Eq. (5). To extract $\dot{\boldsymbol{\omega}}$ from either Eq. (3) or (4), all velocities and accelerations in these closed loop equations must be expressed in terms of the body angular velocity vector. Assume the target attitude is stationary. Then, the MRP kinematic differential equations can be written as are[6–9]

$$\dot{\boldsymbol{\sigma}} = \tfrac{1}{4}[B(\boldsymbol{\sigma})]\boldsymbol{\omega} \tag{7}$$

where the matrix $[B] = [B(\boldsymbol{\sigma})]$ is conveniently expressed as[6,7]

$$[B] = \left[(1 - \boldsymbol{\sigma}^T\boldsymbol{\sigma})I_{3\times 3} + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right] \tag{8}$$

with the skew-symmetric matrix operator being defined in Eq. (6). Differentiating the MRP kinematic differential equation in Eq. (7) we find

$$\ddot{\boldsymbol{\sigma}} = \tfrac{1}{4}[B]\dot{\boldsymbol{\omega}} + \tfrac{1}{4}[\dot{B}]\boldsymbol{\omega} \tag{9}$$

Substituting Eqs. (7) and (9) into the desired linear closed loop dynamics in Eq. (3), the following constraint condition is found.

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} = 0 = \tfrac{1}{4}[B][\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [B]^{-1}([\dot{B}]\boldsymbol{\omega} + 4K\boldsymbol{\sigma})] \tag{10}$$

The following algebra is greatly simplified by making use of the explicit expression of the matrix inverse of $[B]$ given by[25–27]

$$[B]^{-1} = [1/(1 + \sigma^2)^2][B]^T \tag{11}$$

This expression is valid for all nonsingular values of $\boldsymbol{\sigma}$ and can readily verified by using it to confirm that $[B]^{-1}[B] = I_{3\times 3}$. More importantly, the matrix $[B]^{-1}$ is simply a scalar factor multiplied by $[B]^T$. As suggested by Bach in a private communication, yet another interesting property of the $[B]$ matrix is that the MRP Vector $\boldsymbol{\sigma}$ is an eigenvector and $1 + \sigma^2$ an eigenvalue of both $[B]$ and $[B]^T$. By virtue of switching between MRP and the shadow MRP sets, we have $|\boldsymbol{\sigma}| \le 1$, and the matrix $[B]$ is always invertible. As a result, using Eq. (10), the following expression must be true

$$\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [B]^{-1}([\dot{B}]\boldsymbol{\omega} + 4K\boldsymbol{\sigma}) = 0 \tag{12}$$

Eq. (12) yields the necessary $\dot{\boldsymbol{\omega}}$ term to calculate the actual torque vector $\boldsymbol{u}$ in Eq. (5). The vector $\dot{\boldsymbol{\omega}}$ is written as

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - [B]^{-1}([\dot{B}]\boldsymbol{\omega} + 4K\boldsymbol{\sigma}) = \boldsymbol{\phi} \tag{13}$$

where the expression of the right hand side of Eq. (13) is set equal to the new state vector $\boldsymbol{\phi}$. Using the vector product definition of the $[B]$ matrix in Eq. (8), the product $[\dot{B}]\boldsymbol{\omega}$ is expressed as

$$[\dot{B}]\omega = \boldsymbol{\sigma}^T\boldsymbol{\omega}(1 - \sigma^2)\boldsymbol{\omega} - (1 + \sigma^2)(\omega^2/2)\boldsymbol{\sigma} - 2\boldsymbol{\sigma}^T\boldsymbol{\omega}[\tilde{\boldsymbol{\omega}}]\boldsymbol{\sigma} + 2(\boldsymbol{\sigma}^T\boldsymbol{\omega})^2\boldsymbol{\sigma} \tag{14}$$

where the shorthand notation $\omega^2 = \boldsymbol{\omega}^T\boldsymbol{\omega}$ is used. The expression in Eq. (14) is obtained after considerable algebraic manipulations using the identities $[\tilde{\boldsymbol{a}}]\boldsymbol{a} = 0$ and

$$[\tilde{a}][\tilde{a}] = \boldsymbol{a}\boldsymbol{a}^T - \boldsymbol{a}^T\boldsymbol{a}I_{3\times 3}, \qquad \text{any } \boldsymbol{a} \in \mathcal{R}^3 \tag{15}$$

Using this $[\dot{B}]\boldsymbol{\omega}$ expression and Eq. (11) together, we obtain the following

$$[B]^{-1}([\dot{B}]\boldsymbol{\omega} + 4K\boldsymbol{\sigma}) = \left\{\boldsymbol{\omega}\boldsymbol{\omega}^T + [4K/(1 + \sigma^2) - \omega^2/2]I_{3\times 3}\right\}\boldsymbol{\sigma} \tag{16}$$

Making use of this result in Eq. (13), the vector ϕ is finally given by the elegantly simple expression

$$\phi = -P\omega - \left\{\omega\omega^T + [4K/(1+\sigma^2) - \omega^2/2]I_{3\times 3}\right\}\sigma \qquad (17)$$

Therefore the desired Linear Closed Loop Dynamics (LCLD) in Eq. (3) can be rewritten as

$$\ddot{\sigma} + P\dot{\sigma} + K\sigma = \tfrac{1}{4}[B](\dot{\omega} - \phi) = 0 \qquad (18)$$

Substituting $\dot{\omega} = \phi$ into Euler's rotational equations of motion in Eq. (5) yields the required nonlinear feedback control law vector u.

$$u = [\tilde{\omega}][I]\omega + [I]\phi - F_e \qquad (19)$$

We remark that these developments are parallel to those in Ref. 19 where the three vector components of the Euler parameters are used instead of the MRP vector used in this paper. However, it can easily be seen that the singularity at ± 180 degrees is removed by using the MRP vector. It will also be recognized that this control law contains the inertia matrix $[I]$ linearly. When the inertia matrix is unknown, we cannot directly implement Eq. (19). In the following section, we develop an adaptive controller for such situations.

An attractive component of this methodology when dealing with *known* system parameters is that the structure of the closed loop equations can easily be modified using standard linear control theory techniques by appropriate choice of the constants P and K. If it is necessary that the feedback control reject external disturbances, an integral measure of the attitude error is added to the closed loop equations as shown in Eq. (4). Following similar steps as were done previously in this section, the linearizing body angular acceleration vector $\dot{\omega} = \phi$ for closed loop dynamics with an attitude integral measure are written as

$$\phi = -P\omega - \left(\omega\omega^T + \left(\frac{4K}{1+\sigma^2} - \frac{\omega^2}{2}\right)I_{3\times 3}\right)\sigma - 4K_i[B]^{-1}\int_0^t \sigma\,\mathrm{d}t \qquad (20)$$

For this choice of ϕ, the corresponding physical control vector u is of the same form as shown in Eq. (19).

Adaptive Control Formulation

While the vector ϕ is a kinematic quantity depending only on the state vectors σ and ω, to compute the proper linearizing control vector u, the system inertia matrix $[I]$ and the external torque vector F_e must be known precisely. In the following development it is assumed that only very crude estimates of the inertia matrix and external torque vector are known. In this case, the vector ϕ is no longer equal to $\dot{\omega}$, and the actual closed loop dynamics will not be linear.

The following adaptive control law requires that the unknown states appear linearly in the control formulation. Therefore, we rewrite Eq. (19) as

$$u = [L^*]g + [M^*]\phi - F_e^* \qquad (21)$$

where the matrices $[L^*]$ and $[M^*]$ are defined as

$$[L_1] = \begin{bmatrix} 0 & I_{23} & -I_{23} \\ -I_{13} & 0 & I_{13} \\ I_{12} & -I_{12} & 0 \end{bmatrix} \qquad (22)$$

$$[L_2] = \begin{bmatrix} I_{13} & I_{33} - I_{22} & -I_{12} \\ -I_{23} & I_{12} & I_{11} - I_{33} \\ I_{22} - I_{11} & -I_{13} & I_{23} \end{bmatrix} \qquad (23)$$

$$[L^*] \equiv [L_1 \ \vdots \ L_2], \qquad [M^*] \equiv [I] \qquad (24)$$

the vector F_e^* is the true external torque vector, and the 6×1 vector g is defined as

$$g \equiv \begin{bmatrix} \omega_1^2 & \omega_2^2 & \omega_3^2 & \omega_1\omega_2 & \omega_2\omega_3 & \omega_3\omega_1 \end{bmatrix}^T \qquad (25)$$

The control vector expression in Eq. (21) is rewritten by introducing the 3×10 matrix $[Q^*]$

$$[Q^*] = \begin{bmatrix} L^* & \vdots & M^* & \vdots & F_e^* \end{bmatrix} \qquad (26)$$

and the 10×1 state vector x

$$x = \begin{bmatrix} g \\ \phi \\ -1 \end{bmatrix} \qquad (27)$$

into the compact form

$$u = [Q^*]x \qquad (28)$$

Note that Eq. (28) still assumes that all plant parameters are perfectly known. From here on, we assume that the inertia matrix and the external torque vector are not known precisely. The actual control vector u that is implemented is then given by

$$u = [Q(t)]x \qquad (29)$$

where $[Q(t)] = [L(t) \ \vdots \ M(t) \ \vdots \ F_e(t)]$ contains the time-varying adaptive estimates of the unknown system parameters. The difference between the adaptive estimates and true system parameters is expressed through the matrix $[\tilde{Q}]$ as

$$[\tilde{Q}] \equiv [Q(t)] - [Q^*] \qquad (30)$$

Assume that the desired LCLD are to be of the linear PID form given in Eq. (4), then the actual closed loop dynamics, due to the imperfect control vector u in Eq. (29), are found to be

$$\begin{aligned} \ddot{\sigma} + P\dot{\sigma} + K\sigma + K_i\int_0^t \sigma\,\mathrm{d}t &= \frac{1}{4}[B](\dot{\omega} - \phi) \\ &= \frac{1}{4}[B][I]^{-1}\left(-[L^*]g + u + F_e^* - [M^*]\phi\right) \\ &= \frac{1}{4}[B][I]^{-1}([Q(t)]x - [Q^*]x) = \frac{1}{4}[B][I]^{-1}[\tilde{Q}]x \end{aligned} \qquad (31)$$

A key feature of this method is that the desired LCLD *do not depend* on the unknown inertia matrix. This makes it possible to design a desired performance without any knowledge of the actual system parameters.

The goal of the following adaptive control law is to find learning laws for the inertia matrix quantities $[L]$ and $[M]$, and if necessary for the external torque vector F_e, such that the actual closed loop dynamics asymptotically approaches the desired linear form. The main advantage of this control law is that standard linear feedback gain techniques can be employed to find appropriate feedback gains P, K, and K_i that meet system requirements such as control bandwidth and performance. These quantities are typically difficult to enforce with general nonlinear control laws. With the adaptation superimposed on the linearizing control law, we will be guaranteed that the desired closed loop performance is achieved asymptotically, even in the presence of large parameteric uncertainty.

Let the vector σ_r be the solution of the differential equation

$$\ddot{\sigma}_r + P\dot{\sigma}_r + K\sigma_r + K_i\int_0^t \sigma_r\,\mathrm{d}t = 0 \qquad (32)$$

where $\sigma_r(t_0) = \sigma(t_0)$, and $\dot{\sigma}_r(t_0) = \dot{\sigma}(t_0)$. Thus the trajectory $\sigma_r(t)$ represents the desired closed loop performance. Any deviations from this performance are assumed to be due to system model errors $[\tilde{Q}]$. Let the augmented 9×1 state vector ϵ express the difference between the actual states and the reference states.

$$\epsilon = \begin{pmatrix} \int_0^t (\sigma - \sigma_r)\,\mathrm{d}t \\ \sigma - \sigma_r \\ \dot{\sigma} - \dot{\sigma}_r \end{pmatrix} \qquad (33)$$

Using Eq. (31) and (32), note that $\dot{\epsilon}$ is given by

$$\dot{\epsilon} = \underbrace{\begin{bmatrix} 0 & I_{3\times3} & 0 \\ 0 & 0 & I_{3\times3} \\ -K_i I_{3\times3} & -K I_{3\times3} & -P I_{3\times3} \end{bmatrix}}_{[A]} \epsilon + \underbrace{\begin{pmatrix} 0 \\ 0 \\ \boldsymbol{\xi} \end{pmatrix}}_{\boldsymbol{b}} \tag{34}$$

with the vector $\boldsymbol{\xi}$ being defined as

$$\boldsymbol{\xi} = \tfrac{1}{4}[B][I]^{-1}[\tilde{Q}]\boldsymbol{x} \tag{35}$$

We then define the following positive definite Lyapunov function V around the desired reference performance.

$$V = \boldsymbol{\epsilon}^T[S]\boldsymbol{\epsilon} + \mathrm{tr}([\tilde{Q}]^T[\Gamma][\tilde{Q}][\gamma]^{-1}) \tag{36}$$

where $[S]$ and $[\Gamma]$ are yet to be determined positive definite gain matrices, and $[\gamma]$ is a diagonal matrix containing the various learning rates γ_i. Note that the trace operator in Eq. (36) can be written as

$$\mathrm{tr}([\tilde{Q}]^T[\Gamma][\tilde{Q}][\gamma]^{-1}) = \sum_{i=1}^{10} \frac{1}{\gamma_i} \begin{pmatrix} \tilde{Q}_{1i} \\ \tilde{Q}_{2i} \\ \tilde{Q}_{3i} \end{pmatrix}^T [\Gamma] \begin{pmatrix} \tilde{Q}_{1i} \\ \tilde{Q}_{2i} \\ \tilde{Q}_{3i} \end{pmatrix} \tag{37}$$

which is clearly a positive definite function in $[\tilde{Q}]$. Taking the derivative of Eq. (36) and using Eq. (34), we find

$$\dot{V} = \boldsymbol{\epsilon}^T([S][A] + [A]^T[S])\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^T[S]\boldsymbol{b} + 2\mathrm{tr}([\tilde{Q}]^T[\Gamma][\dot{\tilde{Q}}][\gamma]^{-1}) \tag{38}$$

By partitioning the 9×9 matrix $[S]$ into three 9×3 sub-matrices $[S_i]$,

$$[S] = [S_1 \ \vdots \ S_2 \ \vdots \ S_3] \tag{39}$$

the Lyapunov rate $\dot{V}$ is rewritten as

$$\dot{V} = \boldsymbol{\epsilon}^T([S][A] + [A]^T[S])\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^T[S_3]\boldsymbol{\xi} + 2\mathrm{tr}([\tilde{Q}]^T[\Gamma][\dot{\tilde{Q}}][\gamma]^{-1}) \tag{40}$$

Since $[A]$ is a stable matrix, Lyapunov's stability theorem for linear systems states that for any symmetric, positive definite matrix $[R]$, we are guaranteed that there exists a corresponding symmetric, positive definite matrix $[S]$ such that[28]

$$[S][A] + [A]^T[S] = -[R] \tag{41}$$

Therefore, we can pick $[R]$ and numerically solve for a corresponding positive definite matrix $[S]$ for a given stable matrix $[A]$. Using Eqs. (35) and (41), the Lyapunov rate $\dot{V}$ is reduced to

$$\dot{V} = -\boldsymbol{\epsilon}^T[R]\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^T[S_3]\tfrac{1}{4}[B][I]^{-1}[\tilde{Q}]\boldsymbol{x} + 2\mathrm{tr}([\tilde{Q}]^T[\Gamma][\dot{\tilde{Q}}][\gamma]^{-1}) \tag{42}$$

Using several matrix identities listed in Ref. 29, it can be shown that

$$\tfrac{1}{4}\boldsymbol{\epsilon}^T[S_3][B][I]^{-1}[\tilde{Q}]\boldsymbol{x} = \tfrac{1}{4}\mathrm{tr}\left(\boldsymbol{x}^T[\tilde{Q}]^T[I]^{-1}[B]^T[S_3]^T\boldsymbol{\epsilon}\right)$$
$$= \tfrac{1}{4}\mathrm{tr}\left([\tilde{Q}]^T[I]^{-1}[B]^T[S_3]^T\boldsymbol{\epsilon}\boldsymbol{x}^T\right) \tag{43}$$

Using Eq. (43), the Lyapunov rate is expressed as

$$\dot{V} = -\boldsymbol{\epsilon}^T[R]\boldsymbol{\epsilon} + 2\mathrm{tr}\left[[\tilde{Q}]^T\left(\tfrac{1}{4}[I]^{-1}[B]^T[S_3]\boldsymbol{\epsilon}\boldsymbol{x} + [\Gamma][\dot{\tilde{Q}}][\gamma]^{-1}\right)\right] \tag{44}$$

Assuming that the true external torque vector $\boldsymbol{F}_e^*$ is constant, then

$$[\dot{\tilde{Q}}] = [\dot{Q}] - [\dot{Q}^*] = [\dot{Q}] \tag{45}$$

Studying Eq. (44), it is evident that if we set the system parameter learning rate $[\dot{Q}]$ to be

$$[\dot{Q}] = -\tfrac{1}{4}[\Gamma]^{-1}[I]^{-1}[B]^T[S_3]\boldsymbol{\epsilon}\boldsymbol{x}^T[\gamma] \tag{46}$$

the Lyapunov rate function is guaranteed to be of the negative definite form

$$\dot{V} = -\boldsymbol{\epsilon}^T[R]\boldsymbol{\epsilon} \tag{47}$$

Since $\dot{V}$ in Eq. (47) is *negative semidefinite* in the state vector, $\boldsymbol{\epsilon} \in \mathcal{L}_\infty$. The adaptive system parameter estimate errors $[\tilde{L}]$, $[\tilde{M}]$ and $\tilde{\boldsymbol{F}}_e$ are stable. Further, it may be shown that $\boldsymbol{\epsilon} \in \mathcal{L}_2$. Using Barbalat's lemma and classical adaptive control methods,[20] it then follows that $\boldsymbol{\epsilon} \to 0$ as $t \to \infty$. Since the reference motion $\boldsymbol{\sigma}_r(t)$ is globally, asymptotically stable, having $\boldsymbol{\epsilon} \to 0$ implies that the actual closed loop dynamics are also globally, asymptotically stable. Note however that Eq. (46) cannot be implemented directly, because it explicitly depends on the unknown true inertia matrix $[I]$. This problem is circumvented by setting $[\Gamma] = [I]^{-1}$. Selecting this specific $[\Gamma]$ matrix is a crucial step in the stability analysis. Finally, the parameter update law is given by the compact expression

$$[\dot{Q}] = -\tfrac{1}{4}[B]^T[S_3]\boldsymbol{\epsilon}\boldsymbol{x}^T[\gamma] \tag{48}$$

While the inertia matrix adaptive estimate errors $[\tilde{L}]$ and $[\tilde{M}]$ may not necessarily tend to zero, the adaptive external disturbance estimate $\tilde{\boldsymbol{F}}_e$ *will go to zero* if the true external disturbance $\boldsymbol{F}_e^*$ is *constant* in the rotating body frame. This fact can be demonstrated from a standard LaSalle invariance argument, which we omit for the sake of brevity. The relatively simple adaptive learning law in Eq. (48) results in the desired LCLD and the external disturbance being tracked asymptotically *without any apriori* knowledge of either the system inertia matrix or the disturbances themselves. One reason for this result is that the desired LCLD is written as a kinematic expression that does not explicitly depend on any system parameters.

As a practical matter, $\boldsymbol{F}_e^*$ need not be constant to obtain good tracking performance. If $\boldsymbol{F}_e^*$ is large and rapidly varying, certain *robustness* modifications[30] are required to guarantee stability of the adaptive control law, and further tuning may also be required to find practical values for $[\gamma]$, P, K, and K_i.

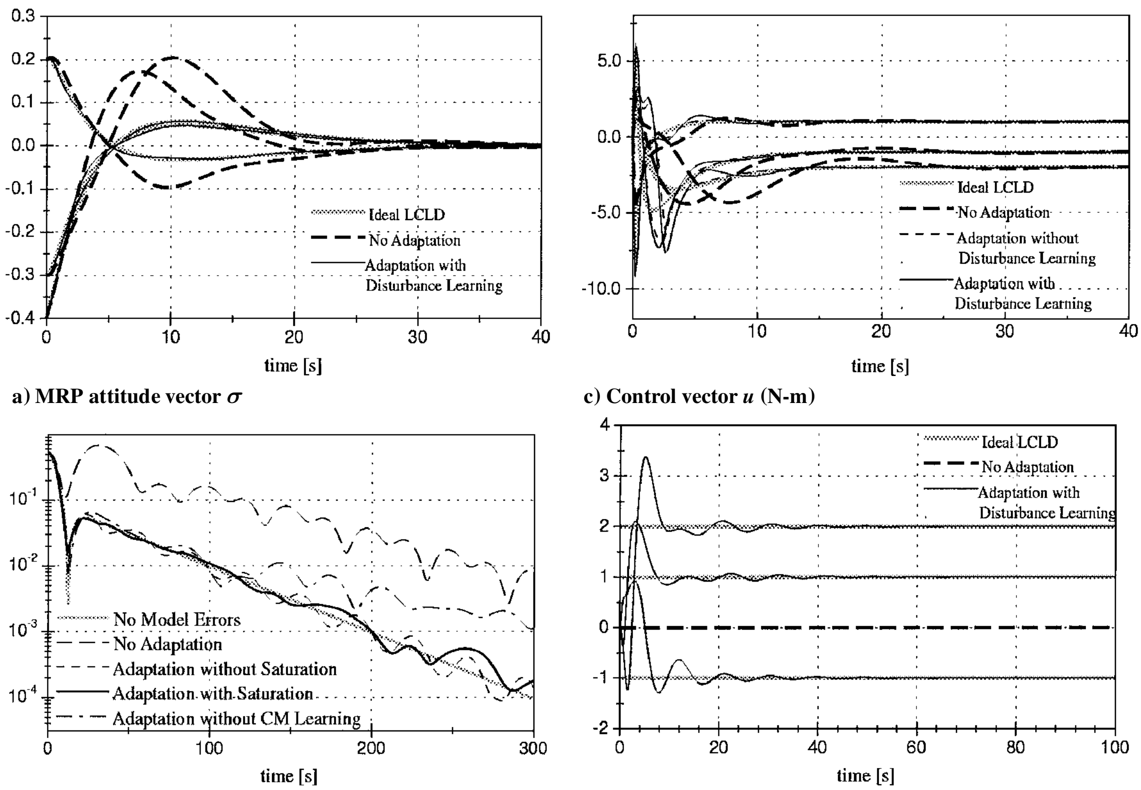
Numerical Simulations

A rigid spacecraft with an initial nonzero attitude and body angular velocity vector is to be brought to rest at a zero attitude vector. The desired LCLD are to be of the PID form shown in Eq. (4) in the presence of large ignorance in the inertia matrix and external disturbance model. The simulation parameters are given in Table 1. The initial $[L(t_0)]$ matrix is constructed out of the corresponding $[M(t_0)]$ matrix elements using Eqs. (22–24).

The scalar learning rates γ_i are all equal with the exception of $\gamma_{F_e} = \gamma_{10}$, which is set to demand a slower external disturbance

Table 1 Numerical simulation parameters

Parameter	Value	Units
$\boldsymbol{\sigma}(t_0)$	$[-0.3 \ \ -0.4 \ \ 0.2]$	
$\boldsymbol{\omega}(t_0)$	$[0.2 \ \ 0.2 \ \ 0.2]$	rad/s
$[I]$	$\begin{bmatrix} 30 & 10 & 5 \\ 10 & 20 & 3 \\ 5 & 3 & 15 \end{bmatrix}$	kg-m^2
$[M(t_0)]$	$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	kg-m^2
K_i	0.090	s^{-3}
K	1.0	s^{-2}
P	3.0	s^{-1}
γ_i	100	
$\gamma_{F_e} = \gamma_{10}$	5	
$\boldsymbol{F}_e^*$	$[2 \ \ 1 \ \ -1]$	N-m
$\boldsymbol{F}_e(t_0)$	$[0 \ \ 0 \ \ 0]$	N-m



a) MRP attitude vector $\boldsymbol{\sigma}$

b) MRP attitude vector magnitude $|\boldsymbol{\sigma}|$

c) Control vector u (N-m)

d) Adaptive external torque estimate (N-m)

Fig. 1 Rigid body stabilization while enforcing LCLD in the presence of large inertia and external disturbance uncertainty.

learning rate than the other γ_i. The positive definite $[R]$ matrix is chosen to be a block-diagonal matrix of the form

$$[R] = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 100\, I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 100\, I_{3\times3} \end{bmatrix} \tag{49}$$

Solving the algebraic Lyapunov equation in Eq. (41) and extracting the third block column matrix, the matrix $[S_3]$ is found to be

$$[S_3] = \begin{bmatrix} 0.055555\, I_{3\times3} \\ 0.702749\, I_{3\times3} \\ 0.400916\, I_{3\times3} \end{bmatrix} \tag{50}$$

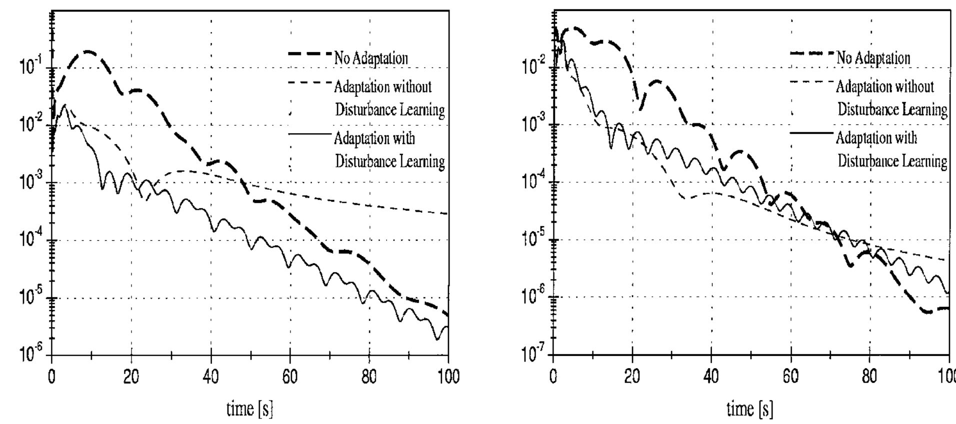
The resulting simulation is illustrated in Fig. 1. The MRP attitude vector components σ_i are shown in Fig. 1a. Without any adaptation, simulations show that the control law based on the nominal inertia matrix $[L(t_0)]$ is still asymptotically stable. However, the transient attitude errors do not match those of the desired LCLD well at all. With adaptation turned on, the performance matches that of the ideal LCLD very closely.

Figure 1b shows the magnitude of the MRP attitude Error vector $\boldsymbol{\sigma}$ on a logarithmic scale. Again the large transient errors of the adaptation-free control law are visible during the first 20 s of the maneuver along with the good final convergence characteristics. The ideal LCLD performance is indicated again through the dotted line. Two versions of the adaptive control law are compared here which differ only by whether or not the external disturbance is adaptively estimated too. On this figure both adaptive laws appear to enforce the desired LCLD very well for the first 40 s of the maneuver. After this the adaptive law without disturbance learning starts to decay at a slower rate, slower even than the no-adaptation solution. Including the external disturbance adaptation clearly improves the final convergence rate. Note however that neither adaptive case starts to deviate from the ideal LCLD case until the MRP attitude error magnitude has decayed to roughly 10^{-3}. Using Eq. (1), this corresponds to having a principal rotation error of roughly 0.23 deg. With external disturbance adaptation, the tracking error at which the LCLD deviations appear is about two orders of magnitude smaller.

The performance of the adaptive control law can be greatly varied by choosing different learning rates. However, since *large* initial inertia matrix and external disturbance model errors are present, the adaptive learning rates were reduced to avoid radical transient torques. The control torque vector components u_i for various cases are shown in Fig. 1c. The no-adaptation torques do not approach the ideal LCLD torque during the transient part of the maneuver. The torques required by either adaptive case are very similar. The difference is that the case with external disturbance learning is causing some extra oscillation of the control about the LCLD case. However, note that with the chosen adaptive learning rates, neither control law exhibits any radical transient torques about the ideal LCLD torque profile. Figure 1d illustrates that the adaptive external disturbance estimate $\boldsymbol{F}_e$ indeed asymptotically approaches the true external disturbance $\boldsymbol{F}_e^*$. By reducing the external disturbance adaptive learning rate γ_{F_e} the transient adaptive estimate errors are kept within a reasonable range.

The purpose of the adaptive control is to enforce the desired LCLD. The previous figures illustrate that the resulting overall system remains asymptotically stable. Figure 2a illustrates the absolute performance error between the actual motion $\boldsymbol{\sigma}(t)$ and the desired linear reference motion $\boldsymbol{\sigma}_r(t)$. This figure demonstrates again the large performance error that results from using the no-adaptation control law with the incorrect system model. Adding adaptation improves the transient performance tracking by up to two orders of magnitude. Without including the external disturbance learning, the final performance error decay rate flattens out. This error will decay to zero. However, with the given learning gains, it does so at a slower rate than if no adaptation were taking place. Adding the external disturbance learning greatly improves the final performance error decay since the system is obtaining an accurate model of the actual constant disturbance. If the initial model estimates were more accurate, then more aggressive adaptive learning rates could be used, resulting in even better LCLD performance tracking. This simulation illustrates though that even in the presence of *large* system uncertainty, it is possible to track the desired LCLD very well.

Figure 2b shows the absolute performance error in attitude rates. Both cases with adaptation added show large reductions in attitude rate errors compared to the the nonadaptive case.



a) Performance error $|\sigma - \sigma_r|$ **b) Performance error $|\dot{\sigma} - \dot{\sigma}_r|$**

Fig. 2 Measure of closed loop controller performance in rigid body stabilization while enforcing LCLD in the presence of large inertia and external disturbance uncertainty.

Conclusion

The feedback linearizing control law presented in terms of the MRP vector $\boldsymbol{\sigma}$ allows for the closed loop dynamics to achieve any desired linear form. Choosing the MRPs as attitude parameters results in a formulation that is globally nonsingular. This choice greatly simplifies the process of finding proper feedback gains P and K which match various performance requirements. To achieve a desired LCLD if modeling errors are present, the ideal case (no uncertainty) control law is augmented with a parameter adaptive update mechanism. For the regulator problem discussed in this paper, this adaptive control tracks the desired LCLD asymptotically and is able to learn a constant external disturbance perfectly. It should be a relatively straightforward exercise to extend the regulator case presented in this paper to the general attitude tracking problem. This control law does not require any previous knowledge of the rigid body inertias or the external disturbance. The law is a direct result of using a *kinematic* differential equation as the desired closed loop dynamics.

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References

[1]Wie, B., and Barba, P., "Quaternion Feedback for Spacecraft Large Angle Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 3, 1985, pp. 360–365.

[2]Wie, B., Weiss, H., and Arapostathis, A., "Quaternion Feedback Regulator Spacecraft Eigenaxis Rotations," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 3, 1989, pp. 375–380.

[3]Dwyer, T. A. W., "Exact Nonlinear Control of Spacecraft Slewing Maneuvers with Internal Momentum Transfer," *Journal of Guidance, Control, and Dynamics*, Vol. 9, No. 2, 1986, pp. 240–247.

[4]Dwyer, T. A. W., "Exact Nonlinear Control of Large Angle Rotational Maneuvers," *IEEE Transactions on Automatic Control*, Vol. AC-29, No. 9, 1984, pp. 769–774.

[5]Slotine, J. E., and Li, W., *Applied Nonlinear Control*, Prentice–Hall, Englewood Cliffs, NJ, 1991, pp. 12–85.

[6]Schaub, H., and Junkins, J. L., "Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters," *Journal of the Astronautical Sciences*, Vol. 44, No. 1, 1996, pp. 1–19.

[7]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[8]Marandi, S. R., and Modi, V. J., "A Preferred Coordinate System and the Associated Orientation Representation in Attitude Dynamics," *Acta Astronautica*, Vol. 15, No. 11, 1987, pp. 833–843.

[9]Tsiotras, P., "Stabilization and Optimality Results for the Attitude Control Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 4, 1996, pp. 772–779.

[10]Robinett, R. D., Parker, G. G., Schaub, H., and Junkins, J. L., "Lyapunov Optimal Saturated Control for Nonlinear Systems," *35th Aerospace Sciences and Meeting*, Reno, NV, 1997, Paper 97-0112.

[11]Junkins, J. L., Akella, M. R., and Robinett, R. D., "Nonlinear Adaptive Control of Spacecraft Maneuvers," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 6, 1997, pp. 1104–1110.

[12]Akella, M., Junkins, J., and Robinett, R., "Structured Model Reference Adaptive Control with Actuator Saturation Limits," *AIAA Guidance and Control Conference*, Boston, 1998, Paper AAS 98-4472.

[13]Schaub, H., Robinett, R. D., and Junkins, J. L., "Globally Stable Feedback Laws for Near-Minimum-Fuel and Near-Minimum-Time Pointing Maneuvers for a Landmark-Tracking Spacecraft," *Journal of the Astronautical Sciences*, Vol. 44, No. 4, 1996, pp. 443–466.

[14]Wen, J. T.-Y., and Kruetz-Delgado, K., "The Attitude Control Problem," *IEEE Transactions on Automatic Control*, Vol. AC-36, No. 10, 1991, pp. 1148–1162.

[15]Joshi, S., Kelkar, A., and Wen, J. T.-Y., "Robust Attitude Stabilization of Spacecraft Using Nonlinear Quaternion Feedback," *IEEE Transactions on Automatic Control*, Vol. AC-40, No. 10, 1995, pp. 1800–1803.

[16]Meyer, G., "On the Use of Euler's Theorem on Rotations for the Synthesis of Attitude Control Systems," NASA TN D-3643, 1966.

[17]Meyer, G., "Design and Global Analysis of Spacecraft Attitude Control Systems," NASA TR R-361, 1971.

[18]Reyhanoglu, M., van der Schaft, A., McClamroch, N., and Kalmanovsky, I., "Dynamics and Control of a Class of Underactuated Mechanical Systems," *IEEE Transactions on Automatic Control*, Vol. AC-44, No. 9, 1999, pp. 1663–1671.

[19]Paielli, R. A., and Bach, R. E., "Attitude Control with Realization of Linear Error Dynamics," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 1, 1993, pp. 182–189.

[20]Narendra, K. S., and Annaswamy, A., *Stable Adaptive Systems*, Prentice–Hall, Englewood Cliffs, NJ, 1989, pp. 18–161.

[21]Sastry, S., and Bodson, M., *Adaptive Control: Stability, Convergence and Robustness*, Prentice–Hall, Upper Saddle River, NJ, 1989, pp. 14–156.

[22]Creamer, G., Delahunt, P., Gates, S., and Levenson, M., "Attitude Determination and Control of Clementine During Lunar Mapping," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 3, 1996, pp. 505–511.

[23]Cristi, R., and Burl, J., "Adaptive Eigenaxis Rotations," *Proceedings of the European Control Conference*, Groningen, The Netherlands, 1993, pp. 243–247.

[24]Ahmed, J., Coppola, V., and Bernstein, D., "Adaptive Asymptotic Tracking of Spacecraft Attitude Motion with Inertia Matrix Identification," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 5, 1998, pp. 684–691.

[25]Schaub, H., "Novel Coordinates for Nonlinear Multibody Motion with Applications to Spacecraft Dynamics and Control," Ph.D Dissertation, Dept. of Aerospace Engineering, Texas A&M Univ., College Station, TX, May 1998.

[26]Milenkovic, V., "Coordinates Suitable for Angular Motion Synthesis in Robots," *Proceedings of Robots VI Conference*, 1982, Paper MS82-217.

[27]Tsiotras, P., "A Passivity Approach to Attitude Stabilization Using Nonredundant Kinematic Parameterizations," *Proceedings of the 34th IEEE Conference on Decision and Control*, 1995, pp. 515–520.

[28]Junkins, J. L., and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures*, AIAA Education Series, Washington DC, 1993, p. 81.

[29]Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, New York, 1977, pp. 353, 354.

[30]Ioannou, P. A., and Sun, J., *Stable and Robust Adaptive Control*, Prentice–Hall, Englewood Cliffs, NJ, 1995, pp. 85–134.